The Double $(\phi'/\phi, 1/\phi)$-Expansion Method and Its Applications

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1. Methodology

In this section, discuss the double $(\phi'/\phi, 1/\phi)$-expansion method. The $(\phi'/\phi, 1/\phi)$-expansion method stands as an effective mathematical tool for establishing analytical soliton solutions to NLEEs of classical or fractional order. The method innately and elegantly provides solutions for a large class of equations. It exhibits advantages over other established techniques. For example, this method simplifies the considered equation by introducing auxiliary functions, ultimately leading to a more manageable, reduced form of the equation. Moreover, the $(\phi'/\phi, 1/\phi)$-expansion method yields solutions that can be revealed as finite assimilation of basic functions. This characteristic proves advantageous for both theoretical analysis and practical applications.

In this part, provide a detailed explanation of the key points of the scheme's sketch, referred to as the $(\phi'/\phi, 1/\phi)$-expansion approach. This approach is utilized to extract a range of solitary wave solutions to the Nizhnik-Novikov-Veselov, nonlinear Schrödinger-Kerr law, unstable nonlinear Schrödinger, Schrödinger-Hirota, Biswas-Arshed, Sasa-Satsuma and Kaup-Newell models. At the outset, consider the second-order equation.

$$\phi''(\xi)+\delta\phi(\xi)=q,$$

(1.1) where δ and q are subjective parameters. We can consider the relationship in the following way:

$$\phi=\phi'/\phi \text{ and } \psi=1/\phi.$$

(1.2) If differentiate equation (1.2) with respect to ξ, the result obtained is as follows:

$$\phi'(\xi)=q\psi(\xi)-\delta-\phi^2(\xi), \psi'(\xi)=-\phi(\xi)\psi(\xi).$$

(1.3) Depending on the value of δ, equation (1.1) have three potential generic solutions.

For type 1 ($\delta<0$): When negative score is allotted to δ, the general solution of Eq. (1.1) will comprise hyperbolic functions in the following form:

$$\phi(\xi)=B_1\sinh(\sqrt{-\delta}\xi)+B_2\cosh(\sqrt{-\delta}\xi)+\frac{q}{\delta}$$

(1.4) together with

$$\psi(\xi)^2=\frac{\delta(\phi^2(\xi)-2q\psi(\xi)+\delta)}{\delta^2 S+q^2},$$

(1.5) whereas B_1, B_2 are arbitrary constants and $S=B_1^2-B_2^2$.

For type 2 ($\delta>0$): When positive score is allotted to δ, the general solution of Eq. (1.1) will comprise trigonometric functions in the following form:

$$\phi(\xi)=B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}$$

(1.6) along with

$$\psi(\xi)^2=\frac{\delta(\phi^2(\xi)-2q\psi(\xi)+\delta)}{\delta^2 S-q^2},$$

(1.7) wherein B_1, B_2 are arbitrary constants and $S=B_1^2+B_2^2$.

Type 3 ($\delta=0$): When δ is equal to zero, the general solution of Eq. (1.1) takes the form of a rational function as follows:

$$\phi(\xi)=\frac{q}{2}\xi^2+B_1\xi+B_2,$$

(1.8) with

$$\psi(\xi)^2=\frac{(\phi^2(\xi)-2q\psi(\xi))}{(B_1^2-2qB_2)},$$

(1.9) where B_1, B_2 are arbitrary constants.

Consider a mathematical model that represents a complex issue and incorporates fractional-order differentiation of a function $\psi=\psi(x,t)$:

$$F(\psi, D_t^{\alpha}\psi, D_t^{2\alpha}\psi, D_x^{2\alpha}\psi, D_{xxt}^{3\alpha}\psi, D_{xxx}^{3\alpha}\psi, ...)=0, 0<\alpha<1,$$

(1.10) wherein F is a polynomial that comprises both ψ and its fractional derivatives. Next, we will outline the fundamental stages of the $(\phi'/\phi, 1/\phi)$-expansion approach for two variables (Odabasi 2020):

1st Step: The following fractional wave transformation:

$$\psi(x,t)=U(\xi), \xi=\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\rho}{v}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}.$$

(1.11) where α represents fractional-order derivative, ξ stipulates wave variable, v signifies wave number, and ρ specifies wave velocity. If the equation (1.10) undergoes the wave transformation (1.11), it results the following nonlinear equation:

$$M(U, U', U'', U''', ...)=0,$$

(1.12) where M represents a polynomial relating U and its associated derivatives.

2nd Step: Following this approach, a polynomial solution depending on the variables ϕ and ψ of Eq. (1.12) can be expressed as:

$$U(\xi)=\sum_{i=0}^{N} a_i\phi^i+\sum_{i=1}^{N} b_i\phi^{i-1}\psi, \phi=\phi'/\phi, \psi=1/\phi,$$

(1.13) where $a_i(i=0,1,...,N)$ and $b_i(i=1,...,N)$ are constants.

3rd Step: The integer N can be determined by following the balancing principle between the nonlinear terms and the derivatives of the highest order in Eq. (1.13).

4th Step: Setting the trial solution from equation (1.13) into equation (1.12), a new equation can be obtained that involves ϕ, ψ and their derivatives. By putting the specific values of ϕ', ψ', and ψ^2, the

higher-order terms of ψ and its derivatives in the resulting new equation can be removed. Now, we equate the coefficients of $\phi^i \psi^j (i=0,1,\ldots,N, j=0,1)$ to zero on both sides of the equation, we obtain a set of algebraic equations.

5th Step: At the end, unraveling this algebraic system of equations leads to the generation of several sets of solutions. These sets of solutions are then utilized in the trial solution (1.13) to derive the necessary analytical traveling wave solutions.

The double $(\phi'/\phi, 1/\phi)$-expansion method is a more versatile and powerful extension of the original (ϕ'/ϕ)-expansion method. While the original method is simpler and suitable for basic FNLEEs, the double method addresses its limitations which offer a broader range of solutions and better handling the complexity of fractional-order systems. The original (ϕ'/ϕ)-expansion method is implemented in standard FNLEEs and is often used as a baseline for studying fractional differential equations in mathematics and physics. In contrast, the double $(\phi'/\phi, 1/\phi)$-expansion method is widely applied to fractional dynamics, optical fiber communication, ion-acoustic models, and fluid mechanics due to its ability to produce precise and diverse solutions. It serves as a valuable and powerful tool for studying FNLEEs in real-world applications. For this purpose, we have used the double $(\phi'/\phi, 1/\phi)$ -expansion method instead of the original (ϕ'/ϕ)-expansion method.

2. Applications

Analytical solutions have several advantages over numerical solutions in various contexts. Here are some key supremacy points of analytical solutions:

They provide precise and exact solutions and offer deeper insights into the underlying mechanisms of problems. They can be more efficient, especially for simpler problems. Despite these advantages, it is important to recognize that closed-form solutions are not always possible. Many complex real-world problems do not have closed-form solutions necessitating the use of numerical methods or approximations. While analytical solutions have significant advantages, it is important to note that they are not always possible for complex or highly nonlinear problems, where numerical solutions become necessary. However, when analytical solutions are available, they provide a powerful tool for understanding and solving problems with precision and insight. Analytical solutions are faster to compute, avoid the need for variable discretization, and can be applied across a wide range of parameters without recomputation. They are stable avoid numerical instability and conditioning issues, and excel in handling complex conditions. Additionally, analytical solutions serve as benchmarks for validating numerical accuracy and contribute to theoretical advancements through rigorous mathematical proofs and symbolic manipulation. Their direct applicability without extensive programming makes them invaluable in fields like physics, engineering, and mathematics where exact solutions are essential for theoretical analysis and problem-solving.

The objective of this part is to use the $(\phi'/\phi, 1/\phi)$-expansion method to obtain a stable broad-ranging general closed-form solutions to the space-time fractional Sasa-Satsuma equation, the space-time fractional Kaup-Newell equation, and the (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov equation from which existing solutions and standard closed-form wave solutions can be extracted.

4.6. The space-time fractional Sasa-Satsuma equation

The space-time fractional Sasa-Satsuma equation (FSSE) with beta fractional derivative is formulated as (Li et al., 2020; Ablowitz et al., 1974; Borg et al., 2024; Wu and Geng, 2017):

$$i D_t^{\alpha}\psi + a D_x^{2\alpha}\psi + b\vee\psi ¿^2\psi + i[\mu D_{xxx}^{3\alpha}\psi + \beta\vee\psi ¿^2 D_x^{\alpha}\psi + \theta D_x^{\alpha}(\vee\psi ¿^2)\psi] = 0.$$

(2.1.1) where $0<\alpha<1$ describing the order of fractional derivative and the wave potential is denoted by $\psi=\psi(x,t)$, where $\psi(x,t)$ is considered to be a complex-valued function of space variable x and temporal variable t. The $D_t^{\alpha}\psi$ and $D_x^{\alpha}\psi$ are the beta fractional derivative. In Eq. (2.1.1), the first term represents the evolution of optical pulses and the coefficients a and b indicate the group velocity dispersion and self-phase modulation respectively. The coefficient μ denotes the third-order dispersion and finally β and θ represent the self-steepening and nonlinear dispersion respectively. We consider the fractional wave transformation

$$\psi(x,t) = U(\xi)e^{i\eta},$$

(2.1.2) where $\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{c}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{d}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$, and c,k,d are arbitrary constants. Substituting the Eq. (2.1.2) into Eq. (2.1.1), cancelling $e^{i\eta}$, and separating the real and imaginary parts, we attain

$$(a+3\mu k)U'' - (d+ak^2+\mu k^3)U + (b+\beta k)U^3 = 0.$$

(2.1.3)

$$\mu U''' - (c+2ak+3\mu k^2)U' + \beta+2\theta U' U^2 = 0.$$

(2.1.4) Integrate once of the Eq. (2.1.4), it is found

$$\mu U'' - (c+2ak+3\mu k^2)U + \frac{\beta+2\theta}{3}U^3 = 0.$$

(2.1.5) Since the Eq. (2.1.3) and Eq. (2.1.4) are polynomials of U, U'', U''', the following conditions are occurred:

$$\frac{a+3\mu k}{\mu} = \frac{d+ak^2+\mu k^3}{c+2ak+3\mu k^2} = \frac{3(b+\beta k)}{\beta+2\theta} = K\,(K\neq 0).$$

(2.1.6) Balancing U'' and U^3, gives $3N=N+2$, so that $N=1$. Thus, the Eq. (2.1.3) has the solution in the form:

$$U(\xi) = a_0 + a_1\varphi(\xi) + b_1\psi(\xi).$$

(2.1.7) where a_0, a_1, b_1 are constants to be determined.

Case 1: For ($\delta<0$), placing Eq. (2.1.7) in (2.1.3) and proceeding with the steps depicted in methodology, several solutions have been acquired for the varied values of δ. There are two sets of solutions which have been expounded below: Set 1: $a_0=0, a_1=\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}, b_1=\frac{i\sqrt{q^2+\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{-2b\delta-2k\delta\beta}}, d=\frac{1}{2}(-2ak^2+a\delta-2k^3\delta+3k\mu\delta)$. Substitution of the above solution set in Eq. (2.1.7) provides

$$U(\xi)=\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}\frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$\frac{+\sqrt{q^2+\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{-2b\delta-2k\delta\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

(2.1.8) Thus, the required exact solution of Eq. (2.1.1), which is generated from Eq. (2.1.2), can be expressed as:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}\frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{\sqrt{q^2+\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{-2b\delta-2k\delta\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.1.9) Here $\eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{d}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{c}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. If we take $B_1=0, B_2\neq 0$ and $q=0$, solution (2.1.9) will be altered to

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a+3\mu k}{2(b+k\beta)}}\tanh\left(\sqrt{-\delta}\,\xi\right)+\frac{\sqrt{(a+3\mu k)}\,\delta\sqrt{S}}{\sqrt{2(b+k\beta)}\,B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.1.10) Similarly, for $B_2=0, B_1\neq 0$ and $q=0$ the solution (2.1.10) takes the form

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a+3\mu k}{2(b+k\beta)}}\coth\left(\sqrt{-\delta}\,\xi\right)+\frac{\sqrt{(a+3\mu k)}\,\delta\sqrt{S}}{\sqrt{2(b+k\beta)}\,B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.1.11) Analogously, other choices of the parametric values provide furthers resourceful and applicable solutions, but for succinctness, these solutions are not documented here.

Set 2: $a_0=0, a_1=-\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}, b_1=\frac{-i\sqrt{q^2+\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{-2b\delta-2k\delta\beta}}, d=\frac{1}{2}\left(-2ak^2+a\delta-2k^3\mu+3k\delta\mu\right)$.

Introducing the above-mentioned solution set 2 in (2.1.7), the acquired solution is

$$U(\xi)=\sqrt{\frac{a+3\mu k}{\sqrt{b+k\beta}}}\frac{\sqrt{\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{\sqrt{2}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

$$\frac{+\sqrt{q^2+\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

(2.1.12) Therefore, Eq. (2.1.2) allowed us to develop the exact solution of Eq. (2.1.1) in the following form:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a+3\mu k}{2\sqrt{b+k\beta}}}\frac{\sqrt{\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right.$$

$$\left.+\frac{\sqrt{q^2+\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.1.13) If, we assign the values $B_1=0, B_2\neq 0$ and $q=0$, the solution (2.1.13) will be transformed into

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{(a+3\mu k)\delta}{2(b+k\beta)}}\tanh\left(\sqrt{-\delta}\,\xi\right)-\frac{\sqrt{(a+3\mu k)\delta}\sqrt{S}}{\sqrt{2(b+k\beta)}B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.1.14) Another form of the solution (2.1.13) can be originated in the same way for $B_1\neq 0, B_2=0$ and $q=0$:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{(a+3\mu k)\delta}{2(b+k\beta)}}\coth\left(\sqrt{-\delta}\,\xi\right)-\frac{\sqrt{(a+3\mu k)\delta}\sqrt{S}}{\sqrt{2(b+k\beta)}B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.1.15)

Case 2: When ($\delta>0$), we attain two sets of possible solutions of the unknown parameters as follows: Set 1: $a_0=0, a_1=\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}, b_1=\frac{i\sqrt{q^2-\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}}, d=\frac{1}{2}\left(a\delta-2k^2\mu-2k^3\mu+3k\delta\mu\right)$. Setting the estimation of the parameters arranged in set 1, equation (2.1.7) provides the result:

$$U(\xi)=\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$\frac{+i\sqrt{q^2-\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}\left\{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}$$

(2.1.16) wherein $\eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{d}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{c}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Hence, the desired solution of Eq. (2.1.16) is derived from transformation of Eq. (2.1.2).

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{-a-3\mu k}{2\sqrt{b+k\beta}}}\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{i\sqrt{q^2-\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}\left\{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.1.17) For $B_1=0, B_2\neq 0$ and $q=0$, the attained result (2.1.17) becomes

$$\psi(x,t)=-\sqrt{\delta}\,e^{i\eta}\left\{\sqrt{\frac{a+3\mu k}{2b+2k\beta}}\tan(\sqrt{\delta}\,\xi)+\frac{\sqrt{\delta S}\sqrt{a+3\mu k}}{B_2\sqrt{2b+2k\beta}}\sec(\sqrt{\delta}\,\xi)\right\}.$$

(1.18) Now, putting $B_1\neq 0, B_2=0$ and $q=0$, solution (2.1.17) yields

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\delta}\sqrt{\frac{-a-3\mu k}{2b+2k\beta}}\cot(\sqrt{\delta}\,\xi)+\frac{\sqrt{\delta S}\sqrt{a+3\mu k}}{B_1\sqrt{2b+2k\beta}}\operatorname{cosec}(\sqrt{\delta}\,\xi)\right\}.$$

(2.1.19) Similar to this, changing the parameter values yields more compatible and useful solutions. However, for brevity, these solutions are not described in full here.

Set 2: $a_0=0, a_1=\sqrt{\frac{-a-3\mu k}{2b+2k\beta}}, b_1=\frac{-i\sqrt{q^2-\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}}, d=\frac{1}{2}(a\delta-2k^2\mu-2k^3\mu+3k\delta\mu).$

Substituting these values in Eq. (2.1.7), we derive

$$U(\xi)=\sqrt{\frac{-a-3\mu k}{2b+2k\beta}}\,\frac{\sqrt{\delta}\left\{B_1\cos(\sqrt{\delta}\,\xi)-B_2\sin(\sqrt{\delta}\,\xi)\right\}}{B_1\cos(\sqrt{\delta}\,\xi)-B_2\sin(\sqrt{\delta}\,\xi)+\frac{q}{\delta}}$$

$$\frac{+\sqrt{q^2-\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}\left\{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

(2.1.20) together with $\eta=\frac{-k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{d}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{c}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. We attain exact soliton solution of Eq. (2.1.20) in exerting transformation (2.1.2) as

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{-a-3\mu k}{2b+2k\beta}}\,\frac{\sqrt{\delta}\left\{B_1\cos(\sqrt{\delta}\,\xi)-B_2\sin(\sqrt{\delta}\,\xi)\right\}}{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}}\right.$$

$$\left.+\frac{\sqrt{q^2-\delta^2 S}\sqrt{a+3\mu k}}{\sqrt{2b\delta+2k\delta\beta}\left\{B_1\sin(\sqrt{\delta}\,\xi)+B_2\cos(\sqrt{\delta}\,\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.1.21) If $B_1=0, B_2\neq 0$ and $q=0$, the required solution of Eq. (2.1.21) is

$$\psi(x,t)=e^{i\eta}\left\{\frac{\sqrt{\delta}\sqrt{-a-3\mu k}}{2b+2k\beta}\tan(\sqrt{\delta}\,\xi)+\frac{\sqrt{\delta S}\sqrt{a+3\mu k}}{B_2\sqrt{2b+2k\beta}}\sec(\sqrt{\delta}\,\xi)\right\}.$$

(2.1.22) If we take $B_1\neq 0, B_2=0$ and $q=0$, solution (2.1.21) will be altered to

$$\psi(x,t)=e^{i\eta}\left\{\frac{-\sqrt{\delta S}\sqrt{-a-3\mu k}}{2b+2k\beta}\cot(\sqrt{\delta}\,\xi)+\frac{\sqrt{\delta S}\sqrt{a+3\mu k}}{B_1\sqrt{2b+2k\beta}}\operatorname{cosec}(\sqrt{\delta}\,\xi)\right\}.$$

(2.1.23)

Case 3: When ($\delta=0$), one set of solution which has been attained in this regard this has been brought out below:

$$a_0=a_0, a_1=a_1, b_1=b_1, d=\frac{b^2(b\mu-a\beta)}{\beta^2}, k=\frac{-b}{\beta}.$$

Putting these values in Eq. (2.1.7), we attain

$$U(\xi)=a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi+2B_2},$$

(2.1.24) Hence, the desired solution of Eq. (2.1.24) is derived from transformation of Eq. (2.1.2).

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi+2B_2}+\frac{2b_1}{\xi^2+2B_1\xi+2B_2}\right).$$

(2.1.25) If we take $B_1=0$ and $B_2\neq 0$, solution (2.1.25) will be altered to

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1\xi}{\xi^2+2B_2}+\frac{2b_1}{\xi^2+2B_2}\right).$$

(2.1.26) Similarly, for $B_1\neq 0$ and $B_2=0$, the solution (2.1.25) takes the form

$$\psi(x,t)=e^{i\eta}\left(a_0+\frac{2a_1(B_1+\xi)}{\xi^2+2B_1\xi}+\frac{2b_1}{\xi^2+2B_1\xi}\right).$$

(2.1.27)

4.7. The space-time fractional nonlinear Kaup-Newell equation

The fractional space-time nonlinear Kaup-Newell equation (Wang, 2024; Mahmood et al., 2024; Ma et al., 2022; Esen et al., 2022; Ahmed et al., 2020) is stated as follows:

$$i D_t^{\alpha}\psi+a D_x^{2\alpha}\psi+ib D_x^{\alpha}(\vee\psi ¿^2\psi)-ic\psi D_x^{\alpha}(\vee\psi ¿^2)=0.$$

(2.2.1) where $\psi=\psi(x,t)$ represents the wave potential and $0<\alpha<1$ indicates the order of the fractional derivative. Here $\psi(x,t)$ is represented the complex-valued function of the space variable x and the temporal variable t. Beta fractional derivatives are $D_t^{\alpha}\psi$ and $D_x^{\alpha}\psi$. Here a represents the coefficient group velocity dispersion and b signifies self-phase modulation. The wave transformation is given below:

$$\psi(x,t)=U(\xi)e^{i\eta},$$

(2.2.2) where $\xi=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. here k, ω, β and v are constants. Putting (2.2.2) into Eq. (2.2.1), it yields a complex equation. Its real part is

$$ak^2U''(\xi)-(v+a\beta^2)U(\xi)-\beta bU^3(\xi)=0,$$

(2.2.3) and imaginary part is

$$(2a\beta k-\omega)U'(\xi)+(3bk-2ck)U^2(\xi)U'(\xi)=0.$$

(2.2.4) We have the following constraint condition of imaginary part (2.2.4) is

$$\omega = 2a\beta k \text{ and } c = \frac{3}{2}b$$

We used the homogeneous balance principle in Eq. (2.2.3) and obtained $N=1$. Thus the trial solution's shape is

$$U(\xi) = a_0 + a_1\varphi(\xi) + b_1\psi(\xi).$$

(2.2.5) where a_0, a_1, b_1 are constants to be calculated.

Case 1: When $\delta<0$: Solution (2.2.5) is placed in equation (2.2.3) and following to the procedure described in methodology section, a coupled of solutions have been achieved for the parameters. Set 1: $a_0=0, a_1=\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}, b_1=\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2+\delta^2 S}}{\delta\sqrt{b\beta}}, k=\sqrt{\frac{\nu(\alpha\beta^2+\nu)}{a\delta}}$ When the solution set assembled above is placed in solution (2.2.5), the following result is obtained:

$$U(\xi) = \left\{ \sqrt{\frac{a\beta^2+\nu}{b\delta\beta}} \frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right) + B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}} \right.$$

$$\left. + \frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2+\delta^2 S}}{\delta\sqrt{b\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}\right\}} \right\}$$

(2.2.6) Consequently, the solution (2.2.2) provides the exact soliton solution to the fractional space-time nonlinear Kaup-Newell equation (2.2.1), which has been presented below:

$$\psi(x,t) = e^{i\eta}\left\{ \sqrt{\frac{a\beta^2+\nu}{b\delta\beta}} \frac{\sqrt{-\delta}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right) + B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}} \right.$$

$$\left. + \frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2+\delta^2 S}}{\delta\sqrt{b\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}\right\}} \right\}$$

(2.2.7) Here $\xi = \frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta = \frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. When $B_1=0, B_2\neq 0$, and $q=0$, the following form of the solution to equation (2.2.7) is obtained:

$$\psi(x,t) = e^{i\eta}\left\{ \sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\tanh\left(\sqrt{-\delta}\,\xi\right) - \frac{i\sqrt{a\beta^2+\nu}\sqrt{\delta S}}{\delta\sqrt{b\beta}\,B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.8) In the same way, if $q=0, B_2=0$, and $B_1\neq 0$, the solution (2.2.7) is as follows:

$$\psi(x,t) = e^{i\eta}\left\{ \sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\coth\left(\sqrt{-\delta}\,\xi\right) - \frac{i\sqrt{S(a\beta^2+\nu)}}{\sqrt{b\beta}\,B_1}\operatorname{cosech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.9) Set 2: $a_0=0, a_1=-\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}, b_1=\frac{-i\sqrt{a\beta^2+\nu}\sqrt{q^2+\delta^2 S}}{\delta\sqrt{b\beta}}, k=\sqrt{\frac{2(\alpha\beta^2+\nu)}{a\delta}}$. Using solution Set 2 in (2.2.5), we get the following solution:

$$U(\xi)=\left\{-\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\frac{\sqrt{-\delta}\left[B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right]}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.-\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2+\delta^2 S}}{\delta\sqrt{b\beta}\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.2.10) Thus, the solution (2.2.2) provides the exact soliton solution to the fractional space-time nonlinear Kaup-Newell equation (2.2.1), which is given below:

$$\psi(x,t)=e^{i\eta}\left\{-\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\frac{\sqrt{-\delta}\left[B_1\cosh\left(\sqrt{-\delta}\,\xi\right)+B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right]}{B_1\sinh\left(\sqrt{-\delta}\,\xi\right)+B_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$\left.-\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2+\delta^2 S}}{m\sqrt{b\beta}\left\{c_1\sinh\left(\sqrt{-\delta}\,\xi\right)+c_2\cosh\left(\sqrt{-\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\right\}$$

(2.2.11) where $\xi=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. If the values for $B_1=0, B_2\neq 0$, and $q=0$ are enforced, the solution (2.2.11) will be converted into

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\tanh\left(\sqrt{-\delta}\,\xi\right)-\frac{i\sqrt{a\beta^2+\nu}\sqrt{\delta S}}{\delta\sqrt{b\beta}\,B_2}\operatorname{sech}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.12) For $B_1\neq 0, B_2=0$, and $q=0$, the solution (2.2.11) can also be obtained in the following way:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\coth\left(\sqrt{-\delta}\,\xi\right)-\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{m\sqrt{b\beta}\,B_1}\operatorname{cosec}\left(\sqrt{-\delta}\,\xi\right)\right\}.$$

(2.2.13)

Case 2: $\delta>0$: In this instance, two sets of parameter values are found. The first set of solutions is described as follows: Set 1: $a_0=0, a_1=\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}, b_1=\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{\delta\sqrt{b\beta}}, k=\sqrt{\frac{2(\alpha\beta^2+\nu)}{a\delta}}$. Using the estimated parameters organized in Set 1 from solution (2.2.5), the solitonic structure solution is derived as:

$$U(\xi)=\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\frac{i\sqrt{\delta}\left[B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right]}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}\right.$$

$$+\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{\delta\sqrt{b\beta}\left\{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\Bigg\}$$

(2.2.14) where $\xi=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Therefore, the solution of Eq. (2.2.1) in relation to the wave variable ξ is derived in the ensuing way:

$$\psi(x,t)=e^{i\eta}\Bigg\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$+\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{m\sqrt{b\beta}\left\{c_1\sin\left(\sqrt{\delta}\,\xi\right)+c_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\Bigg\}$$

(2.2.15) In this case, B_1 and B_2 stand for integral constants, and thus, their values can be freely selected. Therefore, by specifying $B_1=0, B_2\neq 0$, and $q=0$, we obtain the subsequent solution:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\tan\left(\sqrt{\delta}\,\xi\right)+\frac{i\sqrt{S(a\beta^2+\nu)}}{\sqrt{b\beta}\,B_2}\sec\left(\sqrt{\delta}\,\xi\right)\right\}.$$

(2.2.20) When choosing $B_1\neq 0, B_2=0$, and $q=0$, the solution (2.2.19) reduces to:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\cot\left(\sqrt{\delta}\,\xi\right)+\frac{i\sqrt{S(a\beta^2+\nu)}}{\sqrt{b\beta}\,B_1}\operatorname{cosec}\left(\sqrt{\delta}\,\xi\right)\right\}.$$

(2.2.21)

Set 2: $a_0=0, a_1=-\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}, b_1=\frac{-i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{\delta\sqrt{b\beta}}, k=\sqrt{\frac{2(\alpha\beta^2+\nu)}{a\delta}}$. The solution for the solitonic structure is obtained from (2.2.5) by exploiting the assessed parameters arranged in Set 2 as follows:

$$U(\xi)=\Bigg\{-\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\frac{i\sqrt{\delta}\left\{B_1\cos\left(\sqrt{\delta}\,\xi\right)-B_2\sin\left(\sqrt{\delta}\,\xi\right)\right\}}{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}}$$

$$-\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{\delta\sqrt{b\beta}\left\{B_1\sin\left(\sqrt{\delta}\,\xi\right)+B_2\cos\left(\sqrt{\delta}\,\xi\right)+\frac{q}{\delta}\right\}}\Bigg\}$$

(2.2.18) together with $\xi=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. Therefore, the solution for Eq. (2.2.1) with respect to the wave variable ξ is achieved as follows:

$$\psi(x,t)=e^{i\eta}\left\{-\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\frac{i\sqrt{\delta}\left(B_1\cos(\sqrt{\delta}\xi)-B_2\sin(\sqrt{\delta}\xi)\right)}{B_1\sin(\sqrt{\delta}\xi)+B_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}}\right.$$

$$\left.-\frac{i\sqrt{a\beta^2+\nu}\sqrt{q^2-\delta^2 S}}{m\sqrt{b\beta}\left\{c_1\sin(\sqrt{\delta}\xi)+c_2\cos(\sqrt{\delta}\xi)+\frac{q}{\delta}\right\}}\right\}$$

(2.2.19) Here, B_1 and B_2 are integral constants, providing controllability in taking their values. Therefore, when $B_1=0, B_2\neq 0$, and $q=0$, we attain the ensuing solution:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\tan(\sqrt{\delta}\xi)-\frac{i\sqrt{S(a\beta^2+\nu)}}{\sqrt{b\beta}\,B_2}\sec(\sqrt{\delta}\xi)\right\}.$$

(2.2.20) When choosing $B_1\neq 0, B_2=0$, and $q=0$, the solution (2.2.19) reduces to:

$$\psi(x,t)=e^{i\eta}\left\{\sqrt{\frac{a\beta^2+\nu}{b\delta\beta}}\cot(\sqrt{\delta}\xi)+\frac{i\sqrt{S(a\beta^2+\nu)}}{\sqrt{b\beta}\,B_1}\operatorname{cosec}(\sqrt{\delta}\xi)\right\}.$$

(2.2.21)

Case 3: For $\delta=0$: From this perspective, a single set of solutions has been identified as outlined below:

$$a_0=0, a_1=a_1, b_1=2b_1, k=\beta(\alpha\beta-b a_0^2).$$

By substituting these values in (2.2.5), we obtain

$$\psi(\xi)=\frac{2a_1(\eta\xi+B_1)}{q\xi^2+2B_1\xi+2B_2}+\frac{4b_1}{q\xi^2+2B_1\xi+2B_2},$$

(2.2.22) Wherein $\xi=\frac{k}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \eta=\frac{\beta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$. By means of (2.2.22) and (2.2.2), we get the solution of Kaup-Newell equation (2.2.1) given below:

$$\psi(x,t)=e^{i\eta}\left\{\frac{2a_1(q\xi+B_1)}{q\xi^2+2B_1\xi+2B_2}+\frac{4b_1}{q\xi^2+2B_1\xi+2B_2}\right\}.$$

(2.2.23) Moreover, setting $B_1=0$ but $B_2\neq 0$, solution (2.2.23) turns out to be

$$\psi(x,t)=e^{i\eta}\left\{\frac{2a_1q\xi}{q\xi^2+2B_2}+\frac{4b_1}{q\xi^2+2B_2}\right\}.$$

(2.2.24) Similarly, for $B_1\neq 0$ but $B_2=0$, solution (2.2.23) becomes

$$\psi(x,t)=e^{i\eta}\left\{\frac{2a_1(q\xi+B_1)}{q\xi^2+2B_1\xi}+\frac{4b_1}{q\xi^2+2B_1\xi}\right\}.$$

(2.2.25)

4.8. The (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov equation

The ion acoustic model described by the (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov equation (Liu et al., 2020; Zahran et al., 2021; Baskonus et al., 2017; San and Surgin, 2023; Mabenga et al., 2023) is significant in studying nonlinear wave phenomena in plasmas and other ionized media. This equation can be expressed as:

$$D_t^{\alpha}\psi + a(\psi\psi_x) + b(\psi_{xxx} + \psi_{yyy}) + c(\psi_{xyy} + \psi_{xxy}) = 0.$$

(2.3.1) Consider the wave transformation

$$\xi = x + y - \frac{v}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

(2.3.2) where $\psi(x, y, t) = U(\xi)$, (2.3.3) where v is a constant and $0 < \alpha \leq 1$. Using the wave transformation (2.3.2) and (2.3.3) in equation (2.3.1) is remodeled as follows:

$$(2b + 2c)U'' - vU + \frac{a}{2}U^2 = 0.$$

(2.3.4) According to the balancing procedure between U'' and U^2, we attained $N = 2$. Therefore, the trial solution of equation (2.3.4) can be considered as

$$U(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi) + b_1\psi(\xi) + b_2\varphi(\xi)\psi(\xi).$$

(2.3.5) where a_0, a_1, a_2, b_1 and b_2 are constants.

Case 1: When ($\delta < 0$): Solution (2.3.5) is placed in equation (2.3.4) and following the procedure described in the methodology section, a coupled of solutions have been achieved for the parameters.

Set 1: $a_0 = \frac{-8(b+c)\delta}{a}, a_1 = 0, a_2 = \frac{-12(b+c)}{a}, b_1 = \frac{12q(b+c)}{a}, b_2 = \frac{12(b+c)\sqrt{-\delta(\delta^2 S + q^2)}}{a\delta},$

$$v = 2\delta(b+c).$$

Substitution of the above solution set1 in equation (2.3.5) provides,

$$U(\xi) = \frac{8(b+c)\delta}{a} + \frac{12\delta(b+c)\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right) + B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}^2}{a\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}\right\}}$$

$$\frac{+12(b+c)\sqrt{(\delta^2 S + q^2)}\left\{B_1\cosh\left(\sqrt{-\delta}\,\xi\right) + B_2\sinh\left(\sqrt{-\delta}\,\xi\right)\right\}}{a\left\{B_1\sinh\left(\sqrt{-\delta}\,\xi\right) + B_2\cosh\left(\sqrt{-\delta}\,\xi\right) + \frac{q}{\delta}\right\}^2}$$

(2.3.6) where $\xi = x + y - \frac{v}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, v = 2\delta(b+c)$. Therefore, the traveling wave solution in the space-time coordinate of equation is

$$\psi(x,y,t)=\frac{8(b+c)\delta}{a}+\frac{12\delta(b+c)\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}^2}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

$$\frac{+12(b+c)\sqrt{(\delta^2S+q^2)}\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}^2}$$

(2.3.7) In a particular case, setting $B_2=0, B_1\neq 0$, and $q=0$ in (2.3.7), the solitary wave solution turns into

$$\psi(x,y,t)=\frac{8(b+c)\delta}{a}+\frac{12\delta(b+c)}{a}\coth^2(\sqrt{-\delta}\,\xi)$$

$$\frac{+12\delta(b+c)\sqrt{S}}{aB_1}\coth(\sqrt{-\delta}\,\xi)\operatorname{cosech}(\sqrt{-\delta}\,\xi).$$

(2.3.8) Set 2:

$$a_0=\frac{-8(b+c)\delta}{a}, a_1=0, a_2=\frac{-12(b+c)}{a}, b_1=\frac{12q(b+c)}{a}, b_2=\frac{-12(b+c)\sqrt{-\delta(\delta^2S+q^2)}}{a\delta},$$

$$v=2\delta(b+c).$$

The solution for the solitonic structure is obtained from (2.3.5) by exploiting the assessed parameters arranged in set2 as follows:

$$U(\xi)=\frac{-8(b+c)\delta}{a}-\frac{12\delta(b+c)\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}^2}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

$$\frac{+12(b+c)\sqrt{(\delta^2S+q^2)}\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}^2}$$

(2.3.9) where $\xi=x+y-\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, v=2\delta(b+c)$. Therefore, the closed-form wave solution in the space-time coordinates of equation is

$$\psi(x,y,t)=\frac{-8(b+c)\delta}{a}-\frac{12\delta(b+c)\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}^2}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}}$$

$$\frac{+12(b+c)\sqrt{(\delta^2S+q^2)}\left\{B_1\cosh(\sqrt{-\delta}\,\xi)+B_2\sinh(\sqrt{-\delta}\,\xi)\right\}}{a\left\{B_1\sinh(\sqrt{-\delta}\,\xi)+B_2\cosh(\sqrt{-\delta}\,\xi)+\frac{q}{\delta}\right\}^2}$$

(2.3.10) In a particular, setting $B_1=0, B_2\neq 0$, and $q=0$ in (2.3.10), the solitary wave solution turns into

$$\psi(x,y,t)=\frac{8(b+c)\delta}{a}+\frac{12\delta(b+c)}{a}\tanh^2(\sqrt{-\delta}\,\xi)$$

$$\frac{-12\delta(b+c)\sqrt{S}}{aB_2}\tanh(\sqrt{-\delta}\,\xi)\operatorname{sech}(\sqrt{-\delta}\,\xi).$$

(2.3.11)

Case 2: ($\delta>0$): In this instance, two sets of parameters values are found. The first set of solutions is described below: Set 1:

$$a_0=\frac{-8\delta(b+c)}{a}, a_1=0, a_2=\frac{12(b+c)}{a}, b_1=\frac{-12q(b+c)}{a}, b_2=\frac{12(b+c)\sqrt{\delta(\delta^2S-q^2)}}{\delta a},$$

$$v=2\delta(b+c).$$

Inserting the parameters estimated in set 1 into solution (2.3.5), the resulting outcome is as follows:

$$U(\xi)=\frac{-8\delta(b+c)}{a}+\frac{12\delta(b+c)\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}^2}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}}+\frac{12(b+c)\sqrt{(\delta^2S-q^2)}\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

(2.3.12) where $\xi=x+y-\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, v=2\delta(b+c)$. Then the traveling wave solution in the space-time coordinate for equation is

$$\psi(x,y,t)=\frac{-8\delta(b+c)}{a}+\frac{12\delta(b+c)\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}^2}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}}+\frac{12(b+c)\sqrt{(\delta^2S-q^2)}\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

(2.3.13) In a particular case, by setting $B_2=0, B_1\neq 0$, and $q=0$ in (2.3.13), the solitary wave solution

$$\psi(x,y,t)=\frac{-8\delta(b+c)}{a}-\frac{12\delta(b+c)}{a}\cot^2(\xi\sqrt{\delta})$$

$$\frac{+12\delta(b+c)\sqrt{S}}{aB_1}\cot(\xi\sqrt{\delta})\operatorname{cosec}(\xi\sqrt{\delta}).$$

(2.3.14) Set 2:

$$a_0=\frac{8(b+c)\delta}{a},a_1=0,a_2=\frac{-12(b+c)}{a},b_1=\frac{12q(b+c)}{a},b_2=\frac{-12(b+c)\sqrt{\delta(\delta^2S-q^2)}}{\delta a},$$

$$v=2\delta(b+c).$$

By inserting the above estimations into equation (2.3.5), we gather.

$$U(\xi)=\frac{12\delta(b+c)}{a}-\frac{12\delta(b+c)\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}^2}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}}$$

$$\frac{+12(b+c)\sqrt{(\delta^2S-q^2)}\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

(2.3.15) where $\xi=x+y-\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},v=2\delta(b+c)$. Thus, the traveling wave solution in the space-time coordinates of equation is

$$\psi(x,y,t)=\frac{12\delta(b+c)}{a}-\frac{12\delta(b+c)\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}^2}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

$$\frac{+12q(b+c)}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}}$$

$$\frac{+12(b+c)\sqrt{(\delta^2S-q^2)}\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})\right\}}{a\left\{B_1\cos(\xi\sqrt{\delta})-B_2\sin(\xi\sqrt{\delta})+\frac{q}{\delta}\right\}^2}$$

(2.3.16) Moreover, by setting $B_1=0,B_2\neq0$, and $q=0$, in (2.3.16), the solitary wave solution becomes

$$\psi(x,y,t)=\frac{12\delta(b+c)}{a}-\frac{12\delta(b+c)}{a}\tan^2(\xi\sqrt{\delta})$$

$$\frac{+12\delta(b+c)\sqrt{S}}{aB_1}\tan(\xi\sqrt{\delta})\sec(\xi\sqrt{\delta}).$$

(2.3.17)

Case 3: When ($\delta=0$), one set of solution has been attained in this regard this has been brought out below:

$$a_0=\pm\frac{\sqrt{30}}{6}q, a_1=0, a_2=\frac{q}{2\delta}, b_1=\frac{-q^2}{2\delta}, b_2=0, v=\pm\frac{\sqrt{30}}{3}\delta(b+c).$$

By substituting these values in (2.3.5), we obtain

$$U(\xi)=\pm\frac{\sqrt{30}}{6}q+\frac{q}{2\delta}\left(\frac{q\xi+B_1}{\frac{q}{2}\xi^2+B_1\xi+B_2}\right)^2-\frac{q^2}{2\delta\left(\frac{q}{2}\xi^2+B_1\xi+B_2\right)}$$

(2.3.18) where $\xi=x+y-\frac{v}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, v=\pm\frac{\sqrt{30}}{3}\delta(b+c)$. In the space-time coordinate, the wave solution for is attained as

$$\psi(x,y,t)=\pm\frac{\sqrt{30}}{6}q+\frac{q}{2\delta}\left(\frac{q\xi+B_1}{\frac{q}{2}\xi^2+B_1\xi+B_2}\right)^2-\frac{q^2}{2\delta\left(\frac{q}{2}\xi^2+B_1\xi+B_2\right)}$$

(2.3.19) Specifically, by setting $B_2=0$ and $B_1\neq0$ in equation (2.3.19), the wave solution turns out to be

$$\psi(x,y,t)=\pm\frac{\sqrt{30}}{6}q+\frac{q}{2\delta}\left(\frac{q\xi+B_1}{\frac{q}{2}\xi^2+B_1\xi}\right)^2-\frac{q^2}{2\delta\left(\frac{q}{2}\xi^2+B_1\xi\right)}.$$

(2.3.20) Similarly, for $B_1=0$ and $B_2\neq0$, solution (2.3.19) becomes

$$\psi(x,y,t)=\pm\frac{\sqrt{30}}{6}q+\frac{q}{2\delta}\left(\frac{q\xi}{q\xi^2+2B_2}\right)^2-\frac{q^2}{2\delta\left(\frac{q}{2}\xi^2+B_2\right)}.$$

(2.3.21) Similarly, different rich structure soliton solutions can be established for other values of the integral parameters. However, to maintain conciseness, additional solutions beyond those presented here have not been included.

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